

Existence and Stability of Non-Trivial Scalar Field Configurations in Orbifolded Extra Dimensions

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Abstract

We consider the existence and stability of static configurations of a scalar field in a five dimensional spacetime in which the extra spatial dimension is compactified on an S^1/Z_2 orbifold. For a wide class of potentials with multiple minima there exist a finite number of such configurations, with total number depending on the size of the orbifold interval. However, a Sturm-Liouville stability analysis demonstrates that all such configurations with nodes in the interval are unstable. Nodeless static solutions, of which there may be more than one for a given potential, are far more interesting, and we present and prove a powerful general criterion that allows a simple determination of which of these nodeless solutions are stable. We demonstrate our general results by specializing to a number of specific examples, one of which may be analyzed entirely analytically.

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I. INTRODUCTION

The possibility of extra spatial dimensions [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], hidden from our current experiments and observations through compactification or warping, has opened up a wealth of options for particle physics model building and allowed entirely new approaches for addressing cosmological problems.

In many models, standard model fields are supposed to be confined to a submanifold, or brane, while in other models they populate the entire bulk. Common to both approaches, however, is the inclusion of bulk fields beyond pure gravity, either because they are demanded by a more complete theory, such as string theory, or because they are necessary to stabilize the extra dimensional manifold. Thus, a complete understanding of the predictions and allowed phenomenology of extra dimension models necessarily includes a comprehensive consideration of the configurations of these fields.

The allowed configurations of such bulk fields are determined, naturally, by their equations of motion, subject to the boundary conditions imposed by the particular extra-dimensional model under consideration. These might be periodic boundary conditions, in the case of a smooth manifold, or reflection-symmetric ones in the case of an orbifolded extra dimension.

In this paper, building on our recent letter [18] we concern ourselves with a class of allowed nontrivial scalar field configurations [19, 20, 21, 22, 23, 24, 25] in orbifolded extra-dimensional models, neglecting gravity. These configurations exist whenever the potential possesses at least two degenerate minima and we show that they may form a finite tower of kink state solutions. We explicitly demonstrate that all but the lowest-lying of these - the ones with no nodes in the interval - are unstable. In addition we identify a general stability criterion for these lowest-lying states, and provide concrete examples for specific convenient choices of potential.

That a finite tower of nontrivial static configurations may exist, with the possibility of multiple stable ones, allows for new phenomena and constraints on the models, and may have wide-ranging implications for the particle physics and cosmological theories constructed around them [26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43].

We are currently considering the effects of including gravitational effects on the configurations explored in this paper.

II. GENERAL SCALAR POTENTIAL

Since we are neglecting gravity for the entirety of this paper, our background is a flat 4+1 dimensional spacetime, with coordinates $x^M \equiv (x^\mu, y)$, with indices $M, N, \dots = 0, 1, 2, 3, 5$, $\mu, \nu, \dots = 0, 1, 2, 3$. The extra dimension $x^5 \equiv y$ is compactified on an orbifold S_1/Z_2 defined by $y \in (0, \pi R)$, with size πR assumed to be fixed.

Propagating on this background, we consider a real scalar field defined by the action

$$S = \int d^5x \left[\frac{1}{2} \eta^{MN} \partial_M \phi(x, y) \partial_N \phi(x, y) - V(\phi) \right]. \quad (1)$$

Because of the orbifolded geometry, we can demand that the scalar field $\phi(x, y)$ be odd under Z_2 reflections along the extra coordinate (i.e. $\phi(x, y) = -\phi(x, -y)$).

To ensure this, we require that the potential $V(\phi)$ be invariant under the discrete symmetry $\phi \rightarrow -\phi$ and, to simplify notation, we also choose the potential to vanish at $\phi = 0$. We will be particularly interested in potentials which possess multiple degenerate minima, the simplest examples of which are those with two degenerate global minima at $\phi = \pm v$ with $v \neq 0$.

A. Properties of Static Solutions

We seek static field configurations $\phi_A(y)$, parametrized by their amplitudes A , which extremize the action, and with nontrivial y -dependence, subject to the appropriate boundary conditions, namely $\phi_A(0) = 0$ and $\phi_A(\pi R) = 0$.

The field equation satisfied by such solutions is

$$\phi_A'' - \frac{\partial V}{\partial \phi_A} = 0, \quad (2)$$

where a prime denotes a derivative with respect to y . It is easily seen that there exists a first integral, given by

$$\frac{1}{2} \phi_A'^2 + U(\phi_A) = E_A. \quad (3)$$

where $U(\phi) = -V(\phi)$ and E_A is a constant. This choice of nomenclature will be convenient for much of this paper, since it is helpful to think of this problem as that of the position $\phi(y)$ of a particle rolling in time y without friction in the inverted potential $U(\phi)$ [44] (see Figure 1).

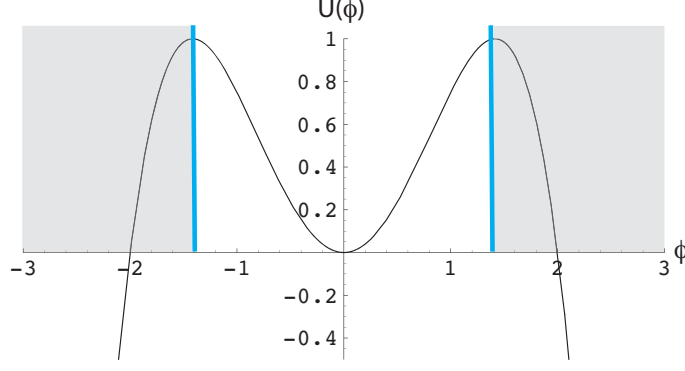


FIG. 1: Mechanical Analogy: Periodic solutions of a particle in the potential $U(\phi) = -V(\phi) = (\mu^2/2)\phi^2 - (\lambda/4)\phi^4$ (here with $\mu^2 = 2$ and $\lambda = 1$) exist when the total energy of the particle lies between $E_{\max} = \frac{\mu^4}{4\lambda}$ (top of the inverted potential) and $E_{\min} = 0$. A particle with energy E_A will undergo a periodic motion of period T , understood as the length of the extra-dimension. Note that this is precisely the potential used in our first example (44).

Since the potential $U(\phi)$ vanishes at $\phi = 0$ and the solution $\phi_A(0)$ also vanishes at $y = 0$, by evaluating (3) at $y = 0$ we see that E is determined by the value of the kink derivative $\phi'_A(y)$ at $y = 0$ via $2E = \phi_A'^2(0)$ (in the mechanical analogy, E is the total energy of the system, which at $y = 0$ is all kinetic energy).

We may rewrite (3) as

$$\frac{1}{2}\phi_A'^2 - V(\phi_A) = -V(A), \quad (4)$$

in which we use the fact that the total energy of the system is equal to the potential energy evaluated at the point where the magnitude of the background solution attains its maximum value, its amplitude A .

Through the mechanical analogy, it is relatively straightforward to see that periodic solutions can only exist for $A < \phi_{\min}$, where ϕ_{\min} is the global maximum of $U(\phi)$ (or the global minimum of $V(\phi)$).

Since the amplitude A parametrizes the different possible nontrivial solutions, it will prove useful to write $\phi_A \equiv \phi_A(y, A)$.

In this notation, we may write the period $T(A)$ of the solution $\phi_A(y, A)$ as

$$T(A) = 2\sqrt{2} \int_0^A \frac{dX}{\sqrt{V(X) - V(A)}}, \quad (5)$$

which must be related to the radius R of the extra dimension.

As noted in [25], the physical size πR of the extra dimension does not need to be equal to the half period $T/2$ of the background solution $\phi_A(y)$, but rather must be a multiple of it

$$2\pi R = (\ell+1) T , \quad (6)$$

with $\ell = 0, 1, \dots, \ell_{\max}$ an integer. Solution(s) with $\ell = 0$ will be nodeless in the interval $(0, \pi R)$, while solutions with $\ell > 0$ will have ℓ nodes between the two boundaries of the orbifold 0 and πR .

Any solution $\phi_A(y)$ that vanishes at the two fixed points of the orbifold $y=0$ and $y=\pi R$, must have vanishing derivative $\phi'_A(y)$ at at least one intermediate value of y .

The symmetry of the potential implies that the points $y = \frac{m_\ell T}{4(\ell+1)}$, where $m_\ell = 1, \dots, 2\ell+1$, are always such special points i.e.

$$\phi'_A \left[\frac{m_\ell T}{4(\ell+1)} \right] = 0 . \quad (7)$$

At all these points, the magnitude of the background solution attains its maximum value A .

Let us now assume that the radius R of the extra dimension is fixed¹. As we have already mentioned and we shall see, there are, in general, multiple nontrivial background solutions corresponding to a given radius, with the precise number depending on the specific choices of the potential $V(\phi)$ and on the radius R . We will identify two classes of solutions among these; background solutions with nodes, and those that are nodeless.

B. Enumerating Solutions

The physical size of the extra dimension is related to the period of the background solution $\phi_A(y)$ by (6).

To see that the maximum number of nodes ℓ_{\max} is finite and provides a lower bound on the maximum number of independent nontrivial solutions, consider the period function $T(A)$ and focus once again on the mechanical analogy. When the potential $U(\phi)$ has a local minimum at $\phi = 0$ (as in Figure 1), it is clear that in the limit $A \rightarrow 0$ the period of a nontrivial solution to Eq. (3) will be determined purely by the quadratic part of the potential $U(\phi)$, i.e. $T(0) = \frac{2\pi}{\mu^2}$, where $\mu^2 = \left. \frac{\partial^2 U}{\partial \phi^2} \right|_{\phi=0}$. This is because ϕ must remain small in that limit

¹ We will assume that some mechanism fixes and stabilizes R without affecting anything else in the setup.

and one can neglect higher order terms in the potential $U(\phi)$ leaving only the quadratic term. In this limit the system becomes a simple harmonic oscillator, with frequency set by the quadratic coefficient of the potential.

As the amplitude A is increased, the period $T(A)$ may increase or decrease, but can never decrease to zero (since the “time” it takes to complete a period can never be zero).

When the potential $U(\phi)$ has a local maximum at $\phi = 0$, nontrivial solutions with an amplitude $A \rightarrow 0$ do not exist. There will be a minimum value of A for which nontrivial solutions exist, and the period $T(A)$ will diverge at that value.

Note also that, if we allow A to approach the value of ϕ at a different local maximum, the period $T(A)$ once again diverges.

Thus, in all cases there exists a global minimum of $T(A)$ that we denote by T_{\min} . For fixed R , nontrivial solutions exist only if $T_{\min} \leq 2\pi R$. It follows that in a size $2\pi R$ there exist at least ℓ_{\max} nontrivial solutions, where

$$\ell_{\max} = IP\left(\frac{2\pi R}{T_{\min}}\right) - 1 \quad (8)$$

and $IP(x) \equiv \text{IntegerPart}(x)$ gives the largest integer less than or equal to x .

A nontrivial solution $\phi_{A,\ell}(y)$ contains ℓ nodes in the orbifold interval. Note that we have now used the number of nodes ℓ along with the amplitude A to parametrize the solutions.

It is, as we shall see explicitly later, possible that there exist two or more solutions with different amplitudes, A_1 and A_2 , say, but with the same period $T(A_1) = T(A_2)$. In particular, since, as we have argued, the function $T(A)$ has a global minimum, if the physical size of the extra dimension is $2\pi R = T(A_1) = T(A_2)$ there exist two nodeless nontrivial kink solutions to our problem.

III. STABILITY OF NONTRIVIAL SOLUTIONS

For a given potential $V(\phi)$ and a given size πR of the orbifold interval, we have shown how to enumerate and construct all possible nontrivial static configurations of our scalar field ϕ . The existence of these configurations is somewhat interesting in its own right, but their physical relevance will depend on their stability properties.

To study this, we begin by adding small perturbations around a given solution $\phi_{A,\ell}(y)$

of (2), writing

$$\phi(x, y) = \phi_{A,\ell}(y) + \varphi(x, y) , \quad (9)$$

where we are again parametrizing the background solution with its amplitude A and its number of nodes ℓ . The 5D Lagrangian then becomes, up to terms quadratic in the perturbations

$$\mathcal{L}^{(5)} = \mathcal{L}_{A,\ell}^{(5)} + \frac{1}{2} \partial^\mu \varphi(x, y) \partial_\mu \varphi(x, y) - \frac{1}{2} \varphi(x, y) \left[-\frac{d^2}{dy^2} + \frac{\partial^2 V}{\partial \phi^2} \Big|_{\phi_{A,\ell}} \right] \varphi(x, y) + .. \quad (10)$$

where $\mathcal{L}_{A,\ell}^{(5)}$ is the lagrangian density corresponding to the background solution.

From this Lagrangian we may obtain the equations of motion of the field $\varphi(x, y)$. Writing $\varphi(x, y) = \varphi_x(x) \varphi_y(y)$ these become

$$\square \varphi_x^n(x) = -M_n^2 \varphi_x^n(x) \quad (11)$$

$$-\varphi_y^{n''}(y) + q(y) \varphi_y^n(y) = M_n^2 \varphi_y^n(y) \quad (12)$$

where

$$q(y) = \frac{\partial^2 V}{\partial \phi^2} \Big|_{\phi_{A,\ell}} . \quad (13)$$

These are the equations of motion of a tower of 4-dimensional scalar fields $\varphi_x^n(x)$ with squared masses M_n^2 and with extra-dimensional profile functions $\varphi_y^n(y)$, which are determined by solving (12).

A. Instability of Solutions with Nodes in the Interval $(0, \pi R)$

A useful result for dealing with those solutions with nodes is obtained by taking the derivative of the equation for the background solution (2), yielding

$$\phi_{A,\ell}'''(y) - \left(\frac{\partial^2 V}{\partial \phi^2} \Big|_{\phi_{A,\ell}(y)} \right) \phi_{A,\ell}'(y) = 0 . \quad (14)$$

Comparing Equations (14) and (12) we see that the derivative $\phi_{A,\ell}'(y)$ of the background solution can be identified as a massless ($M_n^2 = 0$) solution to (12), *but with Neumann boundary conditions rather than the Dirichlet ones we require*².

² This should not come as a big surprise, since it is just the translation mode, the masslessness of which is a reflection of translation symmetry[45, 46]. It cannot be a physical solution since translation invariance is broken in the orbifold.

We now appeal to the general theory of eigenvalues of the Sturm-Liouville problem with Dirichlet (D), periodic (P), semiperiodic (S), and Neumann (N) boundary conditions. That theory contains the following chain of inequalities

$$\begin{aligned}
\lambda_0^N &\leq \lambda_0^P < \lambda_0^S \leq \{\lambda_0^D, \lambda_1^N\} \leq \lambda_1^S < \lambda_1^P \leq \{\lambda_1^D, \lambda_2^N\} \\
&\leq \lambda_2^P < \lambda_2^S \leq \{\lambda_2^D, \lambda_3^N\} \leq \lambda_3^S < \lambda_3^P \leq \{\lambda_3^D, \lambda_4^N\} \\
&\leq \dots,
\end{aligned} \tag{15}$$

relating the towers of eigenvalues corresponding to each different eigensolution $\varphi_i^D, \varphi_i^P, \varphi_i^N$ and φ_i^S defined by each type of boundary condition.

Applying this to any scalar configuration $\phi_{A,\ell}(y)$ with greater than the minimal periodicity ($\ell > 0$), we see that the associated derivative $\phi'_{A,\ell}(y)$, obeying Neumann boundary conditions, will have multiple nodes in the interval $(0, \pi R)$. Thus we may identify it as the eigensolution $\varphi_i^N(y)$, with $i \geq 2$, with its masslessness (from comparing (14) and (12)) implying that the corresponding eigenvalue obeys $\lambda_i^N = 0$.

However (15) implies that $\lambda_2^N > \lambda_0^D$. Therefore, if $\lambda_i^N = 0$ for some $i \geq 2$, then there exists at least one (λ_0^D) eigenvalue of the Dirichlet problem, and possibly more, that are negative!

Thus, *all static solutions with nodes in the interval are unstable.*

B. Stability of Nodeless Solutions

We now turn our attention to the study of perturbations around a solution $\phi_A(y)$ with no nodes in the interval $(0, \pi R = T/2)$ (and therefore parametrized only by the amplitude A). We focus on the sign of the eigenvalue λ of the lowest eigenfunction of equation (12), which we rewrite as

$$\varphi''(y) - [q(y) - \lambda] \varphi(y) = 0, \tag{16}$$

where $\lambda = M_0^2$, and $\varphi(y)$ is the lowest lying eigensolution, which obeys the Dirichlet boundary conditions $\varphi(0) = \varphi(T/2) = 0$.

An important step in our proof of stability will be the study of the massless scalar excitations. We have already identified one such solution, $\varphi_1^N(y) = \phi'_A(y)$, the derivative of the background profile, but it is not a physical one, since it satisfies Neumann boundary

conditions instead of Dirichlet ones. Nevertheless, this solution does allow us to construct a second, linearly independent solution via

$$\varphi_2(y) = \varphi_1^N(y) \int_0^y \frac{ds}{\varphi_1^N(s)^2} = \phi'_A(y) \int_0^y \frac{ds}{\phi'_A(s)^2} \quad (17)$$

This solution automatically satisfies a Dirichlet boundary condition at $y = 0$, but to identify it as a physical solution, we need to establish the circumstances under which it obeys such a condition at $y = T/2$.

In this regard, it is useful to note that equation (16), with $\lambda = 0$, is in the form of the *Hill equation*, for which the following theorem (see, for example [47]) holds
Let $Y_1(t)$ and $Y_2(t)$ be two differentiable solutions of the Hill equation

$$Y''(t) + Q(t) Y(t) = 0, \quad (18)$$

with $Q(t) = Q(t + T/2)$, uniquely determined by the conditions,

$$\begin{aligned} Y_1(0) &= 1, & Y_1(0)' &= 0, \\ Y_2(0) &= 0, & Y_2(0)' &= 1. \end{aligned} \quad (19)$$

When $Q(t) = Q(-t)$ and when $Y_1'(T/2) = 0$ and $Y_1(T/2) = -1$, then

$$Y_2(T/2) = 0 \iff Y_2(T/4)' = 0 \quad (20)$$

This means that, assuming that $Q(t)$ is even, and that the solution $Y_1(t)$ satisfies $Y_1(0) = 1$, $Y_1(T/2) = -1$, $Y_1(T/4) = 0$ and its derivative $Y_1'(y)$ satisfies $Y_1'(0) = Y_1'(T/2) = 0$, then $Y_2(t)$ will obey Dirichlet boundary conditions at $t = 0$ and $t = T/2$, if and only if it obeys a Dirichlet boundary condition at $t = 0$ and a Neumann one at $t = T/4$,

This theorem applies precisely to our problem – equation (16) with $\lambda = 0$, $Q(t) \equiv -q(y)$, and where the function $q(y)$ is even in y due to the symmetry of the potential $V(\phi)$. Thus, we infer that $\varphi_2(y)$ will be a physical solution if and only if it obeys a Dirichlet condition at $y = 0$ and a Neumann one at $y = T/4$. As it turns out, this condition at $y = T/4$ is simpler to study than the one at $T/2$.

Our problem is therefore mapped to that of identifying parameter values for which $\varphi'_2(T/4) = 0$.

Differentiating equation (17) gives

$$\varphi'_2(y) = \phi''_A(y) \int_0^y \frac{ds}{\phi'_A(s)^2} + \frac{1}{\phi'_A(y)}, \quad (21)$$

and so our condition for the existence of a massless scalar excitation is

$$\phi_A''(T/4) \int_0^{T/4} \frac{ds}{\phi_A'^2(s)} + \frac{1}{\phi_A'(T/4)} = 0 . \quad (22)$$

The two terms separately formally diverge, but this divergence must cancel when they are added together.

Now recall our expression (5) for the period $T(A)$ of a solution as a function of the amplitude A . Taking a derivative with respect to A yields

$$\frac{dT}{dA} = \frac{\partial V(A)}{\partial A} \int_0^A \frac{\sqrt{2} dX}{(V(X) - V(A))^{\frac{3}{2}}} + \frac{2\sqrt{2}}{\sqrt{V(X) - V(A)}} \Big|_{X \rightarrow A} , \quad (23)$$

and using (2) and (4) we may rewrite this as

$$\frac{dT}{dA} = 4 \left(\frac{1}{\phi_A'(T/4)} + \phi_A''(T/4) \int_0^{T/4} \frac{dy}{\phi_A'^2(s)} \right) , \quad (24)$$

or simply

$$\frac{dT}{dA} = 4 \varphi_2'(T/4) . \quad (25)$$

Thus, *a massless scalar excitation around a background solution of equation (2), with amplitude A_c , exists if and only if the derivative dT/dA of the period function $T(A)$ vanishes at $A = A_c$. Moreover, this nodeless massless excitation will be the lowest eigenvalue solution of the problem.*

Let us now return to solutions with non-zero eigenvalues. The Rayleigh-Ritz variational result applied to (16) yields an expression for the eigenvalue λ in terms of the eigenfunction $\varphi(y)$

$$\lambda = \frac{\int_0^{T/2} (\varphi'(y)^2 + q(y)\varphi(y)^2) dy}{\int_0^{T/2} \varphi(y)^2 dy} . \quad (26)$$

Because the potential $q(y)$ satisfies $q(y + T/2) = q(y)$ and is symmetric around $y = 0$ and $y = T/4$, it is sufficient to consider the half interval $(0, T/4)$, since the eigenfunctions will be either symmetric or antisymmetric around $y = T/4$.

Thus

$$\lambda = \frac{2}{N} \int_0^{T/4} (\varphi'(y)^2 + q(y) \varphi(y)^2) dy , \quad (27)$$

where $N = 2 \int_0^{T/4} \varphi(y)^2 dy$.

Now assuming that the eigenvalue λ and the eigenfunction φ are differentiable with respect to the amplitude parameter A , one can show that

$$\left. \frac{\partial \lambda}{\partial A} \right|_{A=A_c} = \frac{2}{N} \int_0^{T/4} \varphi^2 \left. \frac{\partial q}{\partial A} \right|_{A=A_c} , \quad (28)$$

where A_c is such that $\left. \frac{dT}{dA} \right|_{A=A_c} = 0$ and therefore is an amplitude for which the lightest scalar excitation around the kink solution is massless.

The variation of $q(y)$ with A can be written as

$$\frac{\partial q}{\partial A} = -\frac{\partial V}{\partial A} \left(\frac{\partial^2 V}{\partial \phi^2} \right)' I(y) , \quad (29)$$

where

$$I(y) = \int_0^y \frac{ds}{\phi'^2(s)} . \quad (30)$$

Thus

$$\left. \frac{\partial \lambda}{\partial A} \right|_{A_c} = -\frac{2}{N} \frac{\partial V}{\partial A} \beta_{A_c} , \quad (31)$$

where we have defined the integral β_{A_c}

$$\beta_{A_c} = \int_0^{T/4} \left(\frac{\partial^2 V}{\partial \phi^2} \right)' \phi'^2 I^3 dy , \quad (32)$$

which may be integrated by parts successively to give

$$\beta_{A_c} = \frac{\partial^2 V}{\partial A^2} \frac{I(T/4)}{\left(\frac{\partial V}{\partial A} \right)^2} - \frac{I(T/4)}{\phi'^2(T/4)} + 3 \int_0^{T/4} \frac{dy}{\phi'^4} . \quad (33)$$

Putting all this together then yields

$$\left. \frac{\partial \lambda}{\partial A} \right|_{A_c} = -\frac{2}{N} \frac{\partial V}{\partial A} \left[\frac{\partial^2 V}{\partial A^2} \frac{I(T/4)}{\left(\frac{\partial V}{\partial A} \right)^2} - \frac{I(T/4)}{\phi'^2(T/4)} + 3 \int_0^{T/4} \frac{dy}{\phi'^4} \right] . \quad (34)$$

To complete the proof, now consider the second derivative of the period function $T(A)$, evaluated at $A = A_c$

$$\left. \frac{d^2 T}{dA^2} \right|_{A_c} = 4 \frac{\partial^2 V}{\partial A^2} I(T/4) - 4 \left(\frac{\partial V}{\partial A} \right)^2 \frac{I(T/4)}{\phi'^2(T/4)} + 12 \left(\frac{\partial V}{\partial A} \right)^2 \int_0^{T/4} \frac{dy}{\phi'^4} . \quad (35)$$

Comparing this to (34) evaluated at $A = A_c$, we obtain our final result

$$\left. \frac{\partial \lambda}{\partial A} \right|_{A_c} = -\frac{1}{2N} \frac{1}{\frac{\partial V}{\partial A}} \left. \frac{d^2 T}{dA^2} \right|_{A_c} . \quad (36)$$

This is our central result, and the proof of stability follows:

- We have demonstrated that at points $A = A_c$ at which $\frac{dT}{dA}$ vanishes, the lightest scalar excitation is massless.
- We have also proved that at $A = A_c$, since $-\frac{\partial V}{\partial A} > 0$, the sign of the derivative with respect to A of the lightest eigenvalue is entirely determined by the sign of $\frac{d^2T}{dA^2}\Big|_{A_c}$.
- This means that for any nontrivial background solution of amplitude A , the sign of λ will be the same as the sign of dT/dA . To see this, consider an interval $A \in (A_{c_1}, A_{c_2})$, over which $T(A)$ is a continuous function of A , and where A_{c_1} and A_{c_2} are two consecutive critical values at which dT/dA vanishes, but at which d^2T/dA^2 is nonzero.

If $dT/dA > 0$ inside that interval (except perhaps at points of inflection) then d^2T/dA^2 is strictly positive at A_{c_1} , and therefore $d\lambda/dA$ is also strictly positive there. Thus in this case λ is positive in the whole interval. If, on the other hand, $dT/dA < 0$ inside the interval, then by an identical argument, λ must also be negative.

It remains to point out that, in the case in which there exists a single critical value of A in a region over which $T(A)$ is continuous, then the above argument still holds, but with the point A_{c_2} replaced by the value of A at which $T(A)$ becomes singular.

We have thus established our general stability criterion: *A static, nodeless solution $\phi_{A_*}(y)$ to equation (2), with amplitude A_* , and period $T(A_*)$, and satisfying $\phi_{A_*}(0) = \phi_{A_*}(T/2) = 0$, is stable if and only if*

$$\frac{dT}{dA}\Big|_{A=A_*} > 0 . \quad (37)$$

Before moving on to some examples, it is worth discussing what happens when we perturb around the trivial solution $\phi_A(y) = 0$. We have seen that there exists a minimal size $\pi R = T_{\min}/2$ of the orbifold interval for which one can find nontrivial static solutions. When the size of the orbifold is smaller than that critical size, the only static background solution possible is the trivial one.

In this case (12) becomes

$$\varphi^{n''}(y) - (\mu^2 - M_n^2) \varphi^n(y) = 0 , \quad (38)$$

where $\mu^2 = \frac{\partial^2 V}{\partial \phi^2}\Big|_{\phi=0}$ may be either positive or negative.

A general solution to this equation is

$$\varphi^n(y) = C \cos \left(\sqrt{M_n^2 - \mu^2} \right) y + D \sin \left(\sqrt{M_n^2 - \mu^2} \right) y . \quad (39)$$

Imposing the Dirichlet boundary condition at $y = 0$ requires that $C = 0$, and imposing the same at $y = \pi R$ yields

$$M_n^2 = \frac{(n+1)^2}{R^2} + \mu^2 . \quad (40)$$

The stability of the trivial solution depends on the sign of M_0^2 and is guaranteed when $\mu^2 > 0$. However, even if $\mu^2 < 0$ the solution will be stable as long as $|\mu^2| < \frac{1}{R^2}$.

This result is quite interesting since it becomes clear now that the trivial solution must be treated carefully given that it can be part of the group of stable static solutions to equation (2). The stability of the trivial solution $\phi_A(y) = 0$ depends on R^2 being smaller than $|\frac{1}{\mu^2}|$, so as long as the potential $V(\phi)$ allows the existence of nontrivial stable kink solutions with period $T < \frac{2\pi}{\mu^2}$, then these solutions will coexist with the trivial solution as the complete set of static classically stable configurations.

IV. ENERGY DENSITY OF NONTRIVIAL CONFIGURATIONS

Since, as we have shown, it is possible for there to exist multiple nodeless and classically stable configurations, we would like to compute the energies of each of these in order to determine the vacuum state of our extra dimensional scalar. This is because quantum mechanical effects will make the highest energy configurations metastable, eventually decaying into the lowest energy configuration, which should then be treated as the true vacuum.

Making explicit once again the dependence on the amplitude parameter A , the energy of a static configuration is

$$E(A) = 2 \int_0^{T_A/2} \left(\frac{1}{2} \phi_A'^2 + V(\phi_A) \right) dy , \quad (41)$$

where again, for simplicity, we assume that $V(0) = 0$.

Using (4) this becomes

$$E(A) = T(A)V(A) + 4\sqrt{2} \int_0^A \sqrt{V(\phi) - V(A)} d\phi , \quad (42)$$

from which we obtain

$$\frac{\partial E}{\partial A} = \frac{\partial T}{\partial A} V(A) \quad (43)$$

where we have used the definition of $T(A)$ in (5).

Now, $V(\phi)$ is always negative when evaluated at an amplitude A at which there exists a nontrivial solution. Also, when such a solution is stable, we have already shown that $\frac{\partial T}{\partial A} > 0$.

Therefore, over any range of A for which $T(A)$ is a continuous function, $\frac{\partial E}{\partial A} < 0$, when evaluated on a stable nontrivial configuration. A possibly interesting corollary of this result is that if one were to consider the equivalent solution on an interval of slightly larger size, this would inevitably have a higher amplitude and therefore a lower energy density. Thus, the energy density of a given solution is lowered by making the interval larger. This may have important ramifications for the stabilization of extra dimensions, which we have not considered here, but are pursuing in other work.

V. EXAMPLES

After this rather general and formal treatment of the stability properties of static solutions, we now turn to some concrete examples with which to better understand the results.

A. Example 1: Mexican-Hat Potential

Our first example is exactly solvable, and the existence of the relevant solutions has been thoroughly studied in [25]. Consider the potential

$$V_1(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\bar{\lambda}}{4}\phi^4, \quad (44)$$

where $[\mu] = [\bar{\lambda}]^{-1} = (\text{Mass})$.

It is easy to see that for $E = \frac{\mu^4}{4\bar{\lambda}}$ one obtains non-trivial solutions known as the kink and anti-kink

$$\phi_{(\text{anti-})\text{kink}}(y) = \pm \frac{\mu}{\sqrt{\bar{\lambda}}} \tanh \left[\frac{\mu}{\sqrt{2}} (y - y_o) \right], \quad (45)$$

where the kink location y_o should be set to zero because of the boundary conditions of the scalar field. This solution interpolates along the (now infinite) extra dimension between the constant background solutions $\phi_{\pm} \equiv \pm \mu/\sqrt{\bar{\lambda}}$.

For $0 < E < \frac{\mu^4}{4\bar{\lambda}}$, we can still integrate (4) to obtain [25]

$$\phi_k(y) = \pm \frac{\mu}{\sqrt{\bar{\lambda}}} \sqrt{\frac{2k^2}{k^2 + 1}} \operatorname{sn} \left(\frac{\mu}{\sqrt{k^2 + 1}} y, k^2 \right), \quad (46)$$

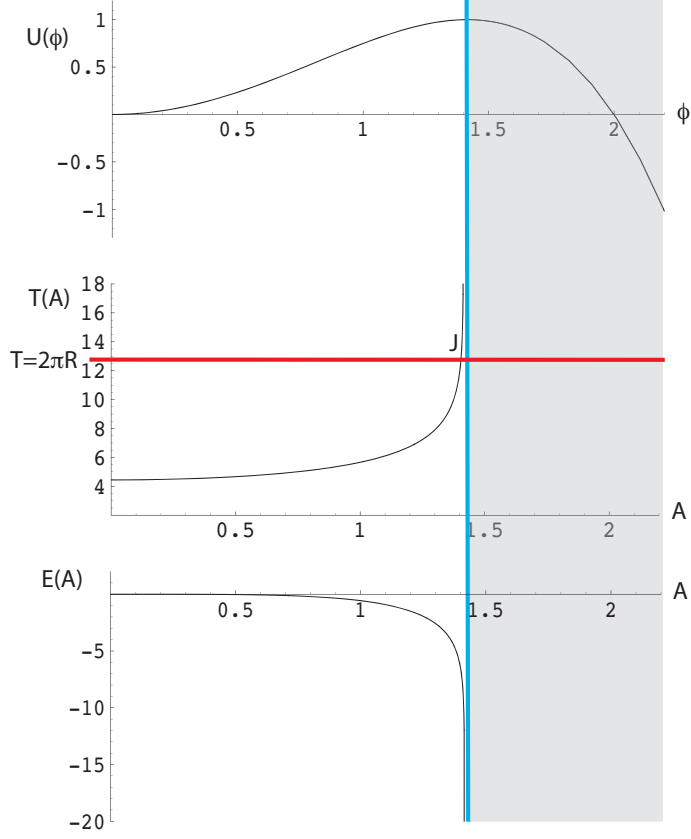


FIG. 2: For this example of the potential, given by (44), we plot the inverted Potential $U_1(\phi) = -V_1(\phi)$, choosing $\mu^2 = 2$ and $\bar{\lambda} = 1$ (top), the period function $T_1(A)$ (middle) and the energy $E_1(A)$ (bottom). There exists a unique, nodeless solutions, here labeled as J , which is stable. In the shaded regions there are no solutions with the appropriate boundary conditions.

where

$$k^2 = \frac{\mu^2 - \sqrt{\mu^4 - 4\bar{\lambda}E}}{\mu^2 + \sqrt{\mu^4 - 4\bar{\lambda}E}} \quad (47)$$

and $\text{sn}(x, k^2)$ is the Jacobi Elliptic Sine-Amplitude, parametrized by the elliptic modulus k (a real parameter such that $0 < k < 1$). Its period is $4K$, where

$$K(k^2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad (48)$$

is the complete elliptic integral of the first kind.

While the above notation provides a natural way to think about this equation, it is convenient (and simple) to rewrite the solutions in terms of the amplitude parameter A , for

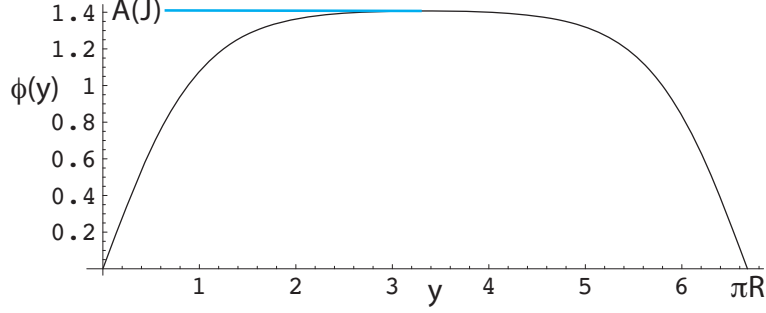


FIG. 3: The single stable solution (point J in figure 2) for the potential (44). Here we have chosen $T_1 = 2\pi R \simeq 13.2$.

consistency with the notation of the previous sections, as

$$\phi_A(y) = A \operatorname{sn} \left(\sqrt{\mu^2 + \frac{\bar{\lambda}}{2} A^2} y, k^2 \equiv \frac{A^2}{2\frac{\mu^2}{\bar{\lambda}} + A^2} \right), \quad (49)$$

where the relationship between the amplitude A and the constant of integration E is

$$A^2 = \frac{\mu^2 - \sqrt{\mu^4 - 4\bar{\lambda}E}}{\bar{\lambda}}. \quad (50)$$

In this example the total number of nontrivial solutions is given by $n_{\max} = \operatorname{IP}(\mu R) - 1$. Since μ is a fixed parameter of the scalar potential and R is the fixed radius of the extra dimension, n_{\max} is completely specified by the model.

The complete set of static nontrivial background solutions consistent with the boundary conditions, for the potential (44) is then

$$\phi_{k_n}(y) = \pm \frac{\mu}{\sqrt{\bar{\lambda}}} \sqrt{\frac{2k_n^2}{k_n^2 + 1}} \operatorname{sn} \left(\frac{\mu}{\sqrt{k_n^2 + 1}} y, k_n^2 \right), \quad (51)$$

where n is an integer such that $0 \leq n \leq n_{\max}$.

The solution with lowest energy, and no nodes in the interval, will be $\phi_{k_0}(y)$ (using k as an equivalent label to A) and is plotted in Fig. (3). The rest of solutions $\phi_{k_n}(y)$ will have nodes and increasing energy. And thanks to our general stability argument they will be unstable.

Note that the radius R of the extra dimension is related to k_0 by

$$2\pi R = \frac{4}{\mu} \sqrt{k_0 + 1} K(k_0^2). \quad (52)$$

In [25] the spectrum and eigenfunctions of the first few scalar excitations around the nodeless background $\phi_{k_0}(y)$ were found. In our case the lowest-lying state is

$$\begin{aligned} \varphi_0(y) = & \operatorname{sn} \left(\sqrt{\mu^2 + \frac{\bar{\lambda}}{2} A^2} y, k^2 \equiv \frac{A^2}{2\frac{\mu^2}{\lambda} + A^2} \right) \\ & \times \operatorname{dn} \left(\sqrt{\mu^2 + \frac{\bar{\lambda}}{2} A^2} y, k^2 \equiv \frac{A^2}{2\frac{\mu^2}{\lambda} + A^2} \right) , \end{aligned} \quad (53)$$

where both sn and dn are Jacobi elliptic functions. The mass eigenvalue of this lowest lying excitation is then given by

$$\lambda \equiv M_0^2 = \frac{3\bar{\lambda}}{2} A^2. \quad (54)$$

which is always positive, demonstrating, as expected, the stability of this solution.

B. Example 2: Distorted Mexican Hat

Even with just two degenerate minima, there exists the possibility for richer structure than in the simple model we have just studied. To see this, consider a second potential

$$V_2(\phi) = -\phi^2 + \frac{5}{26}\phi^4 - \frac{1}{54}\phi^6 + \frac{1}{2000}\phi^8, \quad (55)$$

in which we have set all dimensionful parameters to unity.

As in the previous example, this potential has only two degenerate minima, $\phi = \pm\phi_0$ at which $\frac{\partial V_2}{\partial \phi} = 0$ and $\frac{\partial^2 V_2}{\partial \phi^2} > 0$. However, the crucial difference here is that the second derivative of $V_2(\phi)$ vanishes at two additional field values.

This is enough to allow, for a certain range of choices of πR , the existence of multiple nodeless solutions, illustrated by the points L , M and N on the middle plot of figure 4. Our stability criterion then allows us to immediately conclude that the solutions L and N are stable, while solution M is unstable. These stable solutions are shown in figure 5.

The bottom plot of figure 4 represents the energy of solutions as a function of amplitude and shows that the allowed nodeless solution with higher amplitude (N in this case) has the lower energy.

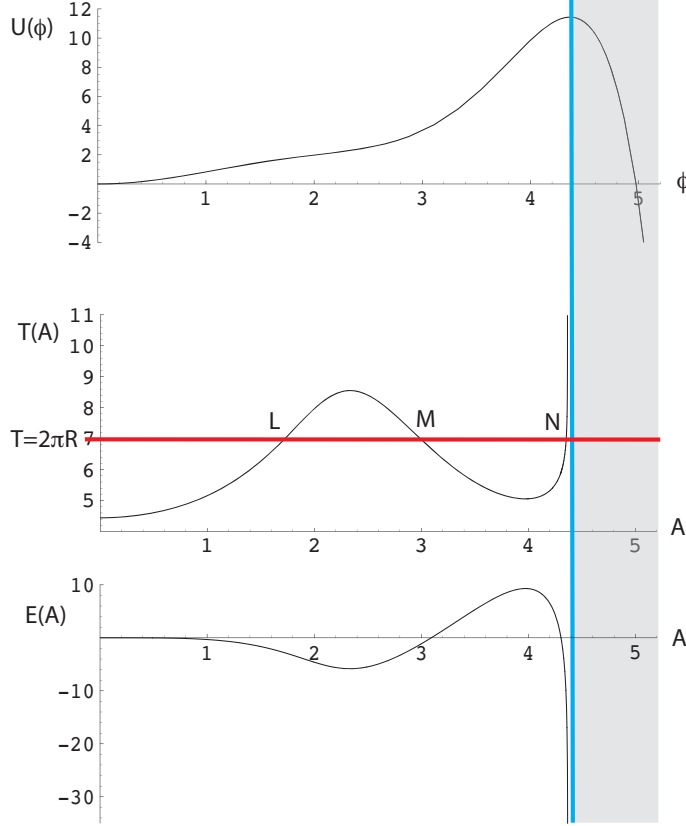


FIG. 4: For this example of the potential, given by (55), we plot the inverted Potential $U_2(\phi) = -V_2(\phi)$ (top), the period function $T_2(A)$ (middle) and the energy $E_2(A)$ (bottom). There exist three distinct nodeless solutions, here labeled as L , M and N , with different values of the amplitude A , but with the same period. The solution at M is unstable, while those at L and N are stable. Further, by integrating (42) we find that N is of lower energy than L . In the shaded regions there are no solutions with the appropriate boundary conditions.

C. Example 3: Many Local Minima

Before concluding, let us provide a more complicated example

$$V_3(\phi) = -\phi^2 - 5\phi^4 + \frac{5}{2}\phi^6 - \frac{1}{3}\phi^8 + \frac{1}{77}\phi^{10}, \quad (56)$$

in which we have set all dimensionful parameters to unity. This potential possesses a pair of degenerate local minima at $\phi = \pm\phi_1$ and a distinct pair of degenerate global minima at $\phi = \pm\phi_2$.

In this case there are two separate intervals of the amplitude A for which there exist nontrivial solutions, as seen in Figure 6. With the choice $2\pi R = 2.6$, the middle plot shows

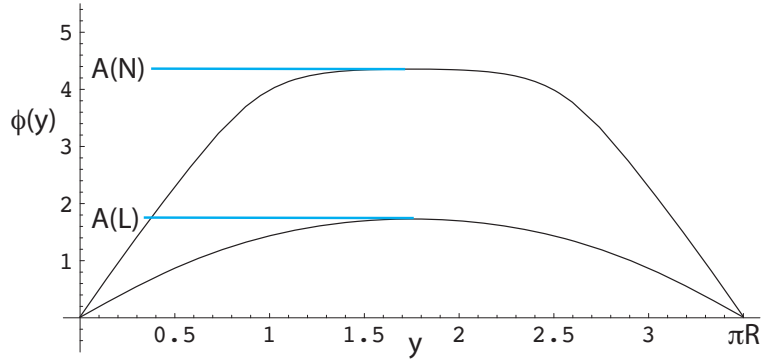


FIG. 5: The two stable solutions (points L and N in figure 4) for the potential (55). The solution at point N , with the larger amplitude, $A(N)$, has the lower energy. Here we have chosen $T_2 = 2\pi R = 7$.

that there are 4 nodeless solutions P , Q , R and S . Only two of them, Q and S , shown in Figure 7, will be stable according to our stability condition and we find that the larger amplitude solution S has the lower energy.

VI. CONCLUSIONS AND OUTLOOK

A thorough understanding of the implications of extra dimensional models requires us to investigate not only perturbative phenomena, but also the allowed distinct background configurations of brane and bulk fields. In infinite dimensions, it is well-known that scalar fields with vacuum manifolds with particular topological properties can give rise to topologically distinct sectors of the theory, characterized by the soliton number. In a compact dimension, the situation is more subtle, since the boundary conditions can affect the stability of configurations identified in the infinite size limit.

In this paper we have studied static, background configurations of scalar fields in constructions in which the bulk space is an S^1/Z_2 orbifold - an interval with reflection-symmetric boundary conditions. We have performed a general stability analysis of such configurations, demonstrating that all solutions with nodes in the interval are unstable. We have also derived a powerful general criterion with which to determine the conditions under which nodeless solutions are stable.

In many cases, there are multiple nodeless solutions, in which case we need to determine which one is the vacuum state of the theory by computing its associated energy density.

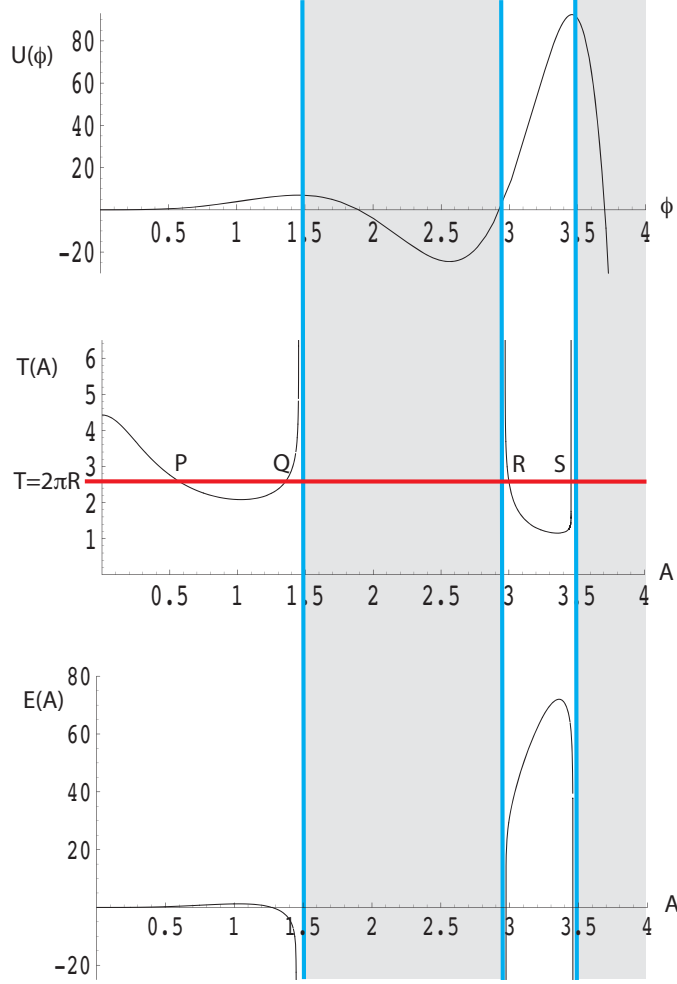


FIG. 6: For this example of the potential, given by (56), we plot the inverted Potential $U_3(\phi) = -V_3(\phi)$ (top), the period function $T_3(A)$ (middle) and the energy $E_3(A)$ (bottom). There exist four distinct nodeless solutions, here labeled as P , Q , R and S , with different values of the amplitude A , but with the same period. Those at P and R are unstable, while those at Q and S are stable. Integrating (42) we find that S is of lower energy than Q . In the shaded regions there are no solutions with the appropriate boundary conditions.

The application of these results to model building and problem solving in extra dimension models may have novel and interesting implications for particle physics, cosmology and the details of stabilization methods. To fully understand such effects will require the inclusion of both quantum effects and gravity, a task that is underway.

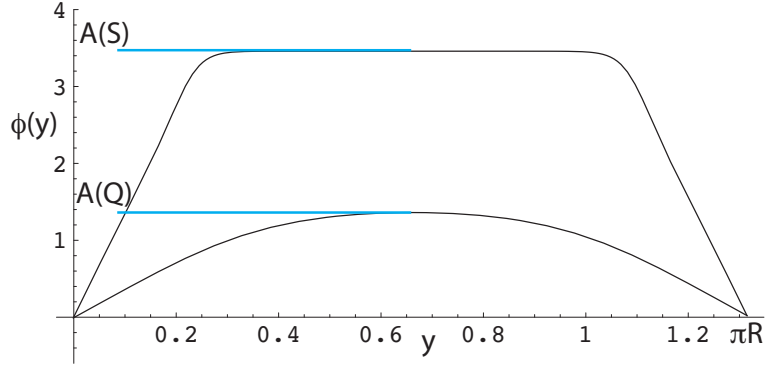


FIG. 7: The two stable solutions (points Q and S in figure 6) for the potential (56). The solution at point S , with the larger amplitude, $A(S)$, has the lower energy. Here we have chosen $T_3 = 2\pi R = 2.6$.

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- [1] T. Kaluza, Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.) **1921**, 966 (1921).
 - [2] O. Klein, Z. Phys. **37**, 895 (1926) [Surveys High Energ. Phys. **5**, 241 (1986)].
 - [3] V. A. Rubakov and M. E. Shaposhnikov, Phys. Lett. B **125** (1983) 136.
 - [4] K. Akama, Lect. Notes Phys. **176**, 267 (1982) [arXiv:hep-th/0001113].
 - [5] I. Antoniadis, Phys. Lett. B **246**, 377 (1990).
 - [6] J. D. Lykken, Phys. Rev. D **54**, 3693 (1996) [arXiv:hep-th/9603133].
 - [7] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **429**, 263 (1998) [arXiv:hep-ph/9803315].
 - [8] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B **436**, 257 (1998) [arXiv:hep-ph/9804398].
 - [9] L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999) 3370 [arXiv:hep-ph/9905221].
 - [10] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 4690 (1999) [arXiv:hep-th/9906064].

- [11] J. Lykken and L. Randall, JHEP **0006**, 014 (2000) [arXiv:hep-th/9908076].
- [12] N. Arkani-Hamed, S. Dimopoulos, G. R. Dvali and N. Kaloper, Phys. Rev. Lett. **84**, 586 (2000) [arXiv:hep-th/9907209].
- [13] I. Antoniadis and K. Benakli, Phys. Lett. B **326**, 69 (1994) [arXiv:hep-th/9310151].
- [14] K. R. Dienes, E. Dudas and T. Gherghetta, Nucl. Phys. B **537**, 47 (1999) [arXiv:hep-ph/9806292].
- [15] N. Kaloper, J. March-Russell, G. D. Starkman and M. Trodden, Phys. Rev. Lett. **85**, 928 (2000) [arXiv:hep-ph/0002001].
- [16] D. Cremades, L. E. Ibanez and F. Marchesano, Nucl. Phys. B **643**, 93 (2002) [arXiv:hep-th/0205074].
- [17] C. Kokorelis, Nucl. Phys. B **677**, 115 (2004) [arXiv:hep-th/0207234].
- [18] M. Toharia and M. Trodden, “Metastable Kinks in the Orbifold”, arXiv:0708.4005 [hep-ph] (2007).
- [19] N. Arkani-Hamed and M. Schmaltz, Phys. Rev. D **61**, 033005 (2000) [arXiv:hep-ph/9903417].
- [20] H. Georgi, A. K. Grant and G. Hailu, Phys. Rev. D **63**, 064027 (2001) [arXiv:hep-ph/0007350],
- [21] D. E. Kaplan and T. M. Tait, JHEP **0111**, 051 (2001) [arXiv:hep-ph/0110126].
- [22] N. S. Manton and T. M. Samols, Phys. Lett. B **207**, 179 (1988).
- [23] M. Sakamoto, M. Tachibana and K. Takenaga, Phys. Lett. B **457**, 33 (1999) [arXiv:hep-th/9902069].
- [24] P. Q. Hung and N. K. Tran, Phys. Rev. D **69**, 064003 (2004) [arXiv:hep-ph/0309115].
- [25] B. Grzadkowski and M. Toharia, Nucl. Phys. B **686**, 165 (2004) [arXiv:hep-ph/0401108].
- [26] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Rev. D **59** (1999) 086004 [arXiv:hep-ph/9807344].
- [27] C. Macesanu and M. Trodden, Phys. Rev. D **71** (2005) 024008 [arXiv:hep-ph/0407231].
- [28] G. D. Starkman, D. Stojkovic and M. Trodden, Phys. Rev. Lett. **87**, 231303 (2001) [arXiv:hep-th/0106143].
- [29] G. D. Starkman, D. Stojkovic and M. Trodden, Phys. Rev. D **63**, 103511 (2001) [arXiv:hep-th/0012226].
- [30] C. Deffayet, G. R. Dvali and G. Gabadadze, arXiv:astro-ph/0106449.
- [31] C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D **65**, 044023 (2002) [arXiv:astro-ph/0105068].

- [32] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B **485**, 208 (2000) [arXiv:hep-th/0005016].
- [33] C. Deffayet, S. J. Landau, J. Raux, M. Zaldarriaga and P. Astier, Phys. Rev. D **66**, 024019 (2002) [arXiv:astro-ph/0201164].
- [34] C. Deffayet, Phys. Lett. B **502**, 199 (2001) [arXiv:hep-th/0010186].
- [35] G. Dvali and M. S. Turner, arXiv:astro-ph/0301510.
- [36] A. Lue, R. Scoccimarro and G. D. Starkman, Phys. Rev. D **69**, 124015 (2004) [arXiv:astro-ph/0401515].
- [37] A. Lue and G. Starkman, Phys. Rev. D **67**, 064002 (2003) [arXiv:astro-ph/0212083].
- [38] P. Binetruy, C. Deffayet, U. Ellwanger and D. Langlois, Phys. Lett. B **477**, 285 (2000) [arXiv:hep-th/9910219].
- [39] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B **615**, 219 (2001) [arXiv:hep-th/0101234].
- [40] P. Binetruy, C. Deffayet and D. Langlois, Nucl. Phys. B **565**, 269 (2000) [arXiv:hep-th/9905012].
- [41] D. J. H. Chung and K. Freese, Phys. Rev. D **61**, 023511 (2000) [arXiv:hep-ph/9906542].
- [42] C. Csaki, M. Graesser, L. Randall and J. Terning, Phys. Rev. D **62** (2000) 045015 [arXiv:hep-ph/9911406].
- [43] J. M. Cline and J. Vinet, JHEP **0202** (2002) 042 [arXiv:hep-th/0201041].
- [44] S. R. Coleman, Phys. Rev. D **15**, 2929 (1977) [Erratum-ibid. D **16**, 1248 (1977)].
- [45] R. Rajaraman, “Solitons and instantons”, North-Holland Publishing Company, 1982
- [46] T. D. Lee and Y. Pang, Phys. Rept. **221**, 251 (1992).
- [47] W. Magnus, S. Winkler, “Hill’s Equation”, Interscience Publishers, 1966